Section A (40 marks)

1. This is a question on polynomials. Let $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

(a) If a is a root of the equation $f(x) = 0$, find a^7. (2 marks)
(b) Write down, in polar form, the six distinct roots of the equation $f(x) = 0$. (1 mark)
(c) Find the remainder when $f(x^7)$ is divided by $f(x)$. (4 marks)

2. This is a question on the properties of definite integrals.

(a) If $f(x)$ is an integrable periodic function with period p, prove that $\int_0^p f(x)dx = \int_{\frac{-p}{2}}^{\frac{p}{2}} f(x)dx$ and $\int_a^{a+p} f(x)dx = \int_0^p f(x)dx$. (4 marks)
(b) Show that if $g(x)$ is an integrable periodic odd function with period p, then $\int_a^{a+kp} g(x)dx = 0$, where k is a positive integer. (3 marks)

3. This is a question on the application of definite integration. Let there be a region bounded by the curve $y = x\ln(x)$, the straight line $x = e$, and the x-axis.

(a) Find the area of the bounded region. (3 marks)
(b) Find the volume of the solid of revolution when the bounded region is revolved about the x-axis. (4 marks)

4. This is a question on sequences. Let $a_1 = \frac{3}{2}$, $a_2 = \frac{7}{12}$ and $6a_{n+2} = 5a_{n+1} - a_n$ for all positive integers n.

(a) Using mathematical induction, prove that $a_n = \frac{1}{2^n} + \frac{1}{3^{n-1}}$ for any positive integer n. (4 marks)
(b) Does there exist a positive integer m such that $\sum_{k=1}^m a_k > 3$? Explain your answer. (3 marks)

5. This is a question on sequences. Let $S = \sum_{k=1}^n (1 + \frac{1}{k})$, where $n \in \mathbf{N} \setminus \{1\}$.

(a) Using A.M. $\geq$ G.M., or otherwise, prove that $\frac{2n-S}{n-1} \geq (\frac{1}{n})^{\frac{1}{n-1}}$. (3 marks)
(b) Prove that $2n - (n-1)n^{\frac{1}{1-n}} \geq S \geq n(n+1)^{\frac{1}{n}}$. (3 marks)

6. This is a question on continuity and differentiability. It is given that $f : \mathbf{R} \to \mathbf{R}$ is a differentiable function satisfying $f(\pi) = -1$ and $f'(\pi) = 3$. Let k be a real constant and $g : \mathbf{R} \to \mathbf{R}$ be defined by

$$g(x) = \begin{cases} f(x) + x + k & \text{when } x \leq \pi, \\ \frac{\sin x}{x-\pi} & \text{when } x > \pi. \end{cases}$$

Suppose that $g(x)$ is continuous at $x = \pi$.

(a) Find k. (2 marks)
(b) Is $g(x)$ differentiable at $x = \pi$? Explain your answer. (4 marks)

END OF SECTION A

Section B (60 marks)

7. This is a question on graph plotting. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{10(x+2)}{x^2+5}$.

(a) Find $f'(x)$ and $f"(x)$. (2 marks)
(b) Solve $f'(x) > 0$ and $f''(x) > 0$. (2 marks)
(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y = f(x)$. (3 marks)
(d) Find the asymptote(s) of $y = f(x)$. (3 marks)
(e) Sketch the graph of $y = f(x)$. (3 marks)
(f) Sketch the graph of $y = f(|x - 1|)$. (2 marks)

8. This is a question on properties of functions and differentiability. It is given that $f : \mathbf{R} \to \mathbf{R}$ satisfies the following conditions:
(1) $f(x + y) = f(x)f(y) - f(x) - f(y) + 2$ for all $x, y \in \mathbf{R}$;
(2) there exists a unique real number r such that $f(r) = 2$.

(a) Prove that $f(0) = 2$. (3 marks)
(b) Is f a injective function? Explain your answer. (3 marks)
(c) Is f a surjective function? Explain your answer. (3 marks)
(d) Suppose that $\lim_{h \to 0} \frac{f(h)-2}{h} = 12$.
(i) Prove that f is differentiable everywhere and $f'(x) = 12f(x) - 12$ for all $x \in \mathbf{R}$. (3 marks)
(ii) By differentiating $e^{-12x}f(x)$, find $f(x)$. (3 marks)

9. This is a question on definite integral and limits of sequences.

(a) For each positive integer n, let $I_n = \int_0^\pi e^{-x}(\pi - x)^n dx$.
(i) Evaluate I_1. (2 marks)
(ii) Express I_{n+1} in terms of I_n. (1 mark)
(iii) Prove that $\sum_{k=0}^{n}(-1)^k \frac{\pi^k}{k!} = (-1)^n \frac{I_n}{n!} + e^{-\pi}$. (3 marks)
(b) For each positive integer n, let $a_n = \frac{\pi^n}{n!}$.
(i) Prove that $a_{n+1} < a_n$ for all $n > 3$. (2 marks)
(ii) Using (b)(i), or otherwise, prove that $\lim_{n\to\infty} a_n$ exists. Also evaluate $\lim_{n\to\infty} a_n$. (3 marks)
(c) Using (a)(iii), evaluate $\sum_{k=0}^{\infty}(-1)^k \frac{\pi^k}{k!}$. (4 marks)

10. This is a question on Mean Value Theorem.

(a) Denote the closed interval $[1,2]$ and the open interval (1,2) by $\mathbf{I}$ and $\mathbf{J}$ respectively.
(i) Assume that real-valued functions p and q are continuous on $\mathbf{I}$ and $q(x) > 0$ for all $x \in \mathbf{J}$. Define $h(x) = \int_1^2 q(t)dt \int_1^x p(t)q(t)dt - \int_1^2 p(t)q(t)dt \int_1^x q(t)dt$ for all $x \in \mathbf{I}$.
(1) Find $h'(x)$ for all $x \in \mathbf{J}$. (1 mark)
(2) Using the result of (a)(i)(1) and Mean Value Theorem to prove that there exists $\beta \in \mathbf{J}$ such that $\int_1^2 p(x)q(x)dx = p(\beta)\int_1^2 q(x)dx$. (4 marks)
(ii) Let f and g be real-valued functions such that f' and g' are continuous on $\mathbf{I}$ and $f'(x) > 0$ for all $x \in \mathbf{J}$. Prove that there exists $c \in \mathbf{J}$ such that
$\int_1^2 f(x)g'(x)dx = f(2)g(2) - f(1)g(1) - g(c)(f(2) - f(1))$. (4 marks)
(b)
(i) Find $\frac{d}{dx}\cos x^{100}$. (1 mark)
(ii) Using (a)(ii), prove that $|\int_1^2 \sin x^{100} dx| \leq \frac{1}{50}$. (5 marks)

11. This is a question on series and sequences.

(a) Let $\lambda > 1$. Prove that $(1+x)^{\lambda} > 1 + \lambda x$ for any $x > 0$. (3 marks)

(b) For any positive integer n, define $a_n = (1+\frac{1}{n})^n$ and $b_n = (1+\frac{1}{n})^{n+1}$.

(i) Using (a), or otherwise, prove that $a_{n+1} > a_n$. (2 marks)

(ii) Prove that $\frac{b_n}{b_{n+1}} = (1 + \frac{1}{n(n_+2)})^{n+1}(\frac{n+1}{n+2}) > 1$. (3 marks)

(iii) Using (b)(i) and (b)(ii)(2), prove that both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. (2 marks)

(iv) Find $\prod_{k=1}^{n} a_k$ and $\prod_{k=1}^{n} b_k$. Hence prove that $(n+1)^{n+1} > n!e^n > (n+1)^n$, where $e = \lim_{n\to\infty} a_n$. (5 marks)

END OF PAPER